An exponential functional of random walks

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Abstract

The aim of this paper is to investigate discrete approximations of the exponential functional $\int_0^\infty \exp(B(t) - \nu t)\, dt$ of Brownian motion (which plays an important role in Asian options of financial mathematics) by the help of simple, symmetric random walks. In some applications the discrete model could be even more natural than the continuous one. The properties of the discrete exponential functional are rather different from the continuous one: typically its distribution is singular w.r.t. Lebesgue measure, all of its positive integer moments are finite and they characterize the distribution. On the other hand, using suitable random walk approximations to Brownian motion, the resulting discrete exponential functionals converge a.s. to the exponential functional of Brownian motion, hence their limit distribution is the same as in the continuous case, namely, the one of the reciprocal of a gamma random variable, so absolutely continuous w.r.t. Lebesgue measure. This way we give a new, elementary proof for an earlier result by Dufresne and Yor as well.

1 Introduction

The geometric Brownian motion (originally introduced by the economist P. Samuelson in 1965) plays a fundamental role in the Black–Scholes theory of option pricing, modeling the price process of a stock. It can be explicitly given in terms of Brownian motion (BM) B as

$$S(t) = S_0 \exp\left(\sigma B(t) + \left(\mu - \sigma^2/2\right) t\right), \qquad t \geq 0.$$

In the case of Asian options one is interested in the average price process

$$A(t) = \frac{1}{t} \int_0^t S(u)\, du, \qquad t \geq 0.$$

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The following interesting result is true for the distribution of a closely related, widely investigated exponential functional of BM:

$$\mathcal{I} = \int_0^\infty \exp(B(t) - \nu t)\, dt \stackrel{\mathsf{d}}{=} \frac{2}{Z_{2\nu}} \qquad (\nu > 0). \tag{1}$$

Here $Z_{2\nu}$ is a gamma distributed random variable with index 2ν and parameter 1, while $\stackrel{\mathsf{d}}{=}$ denotes equality in distribution. This result was proved by [Dufresne (1990)] using discrete approximations with gamma distributed random variables and also by [Yor (1992)], using rather ingenious stochastic analysis tools. For more background information see [Yor (2001)] and [Csörgő, M. (1999)].

As a consequence, the pth integer moment of $\mathcal{I}$ is finite iff $p < 2\nu$ and

$$\mathbf{E}(\mathcal{I}^p) = 2^p \frac{\Gamma(2\nu - p)}{\Gamma(2\nu)}. \tag{2}$$

On the other hand, all negative integer moments, also given by (2), are finite and they characterize the distribution of $\mathcal{I}$.

The situation is much nicer when BM with negative drift is replaced in the model by the negative of a subordinator $(\alpha_t, t \geq 0)$, that is, by the negative of a non-decreasing process with stationary and independent increments, starting from the origin. Then, as was shown by [Carmona et al. (1997)], all positive integer moments of $\mathcal{J} = \int_0^\infty \exp(-\alpha_t)\, dt$ are finite:

$$\mathbf{E}(\mathcal{J}^p) = \frac{p!}{\Phi(1)\cdots\Phi(p)}, \qquad \Phi(\lambda) = -\frac{1}{t}\log \mathbf{E}(\exp(-\lambda\alpha_t)), \tag{3}$$

and in this case the positive integer moments characterize the distribution of $\mathcal{J}$.

To achieve a similar favorable situation in the BM case, at least in an approximate sense, it is a natural idea to use a simple, symmetric random walk (RW) as an approximation, with a large enough negative drift. Besides, in some applications a discrete model could be more natural than a continuous one. It seems important that, as we shall see below, the discrete case is rather different from the continuous case in many respects.

So let $(X_j)_{j=1}^\infty$ be an i.i.d. sequence with $\mathbf{P}(X_1 = \pm 1) = \frac{1}{2}$ and $S_0 = 0$, $S_k = \sum_{j=1}^k X_j$ $(k \geq 1)$. Introduce the following approximation of $\mathcal{I}$:

$$Y = \sum_{k=0}^\infty \exp(S_k - k\nu) = 1 + \xi_1 + \xi_1\xi_2 + \cdots, \qquad \xi_j = \exp(X_j - \nu), \tag{4}$$

where $\nu > 0$. In this paper we investigate the properties of Y, which will be called the discrete exponential functional of the given RW, or shortly, the discrete exponential functional.

In Section 2 below it turns out that the distribution of Y is singular w.r.t. Lebesgue measure if $\nu > 1$. Then in Section 3 we find a formula, similar to (2), for all positive integer moments of Y when $\nu > 1$, and, because Y is bounded then, these moments really characterize its distribution. Finally, in Section 4 we use a nested sequence of RWs to obtain a.s. converging approximations of $\mathcal{I}$, and this way an elementary proof of result (1) of Dufresne and Yor as well.

2 The distribution of the discrete exponential functional

Let us start with a natural generalization: $(\xi_j)_{j=1}^{\infty}$ be i.i.d., $\xi_j > 0$. Consider first the finite polynomial

$$\begin{aligned} Y_n &= 1 + \xi_1 + \xi_1\xi_2 + \cdots + \xi_1 \cdots \xi_n \\ &= 1 + \xi_1(1 + \xi_2 + \xi_2\xi_3 + \cdots + \xi_2 \cdots \xi_n) \qquad (n \geq 1), \end{aligned} \tag{5}$$

$Y_0 = 1$. This implies the following equality in distribution:

$$Y_n \stackrel{\mathsf{d}}{=} 1 + \xi Y_{n-1}, \tag{6}$$

where $\xi \stackrel{\mathsf{d}}{=} \xi_1$, and ξ is independent of Y_{n-1}. Since $Y_n \nearrow Y = 1 + \xi_1 + \xi_1\xi_2 + \cdots$ a.s., we get the basic *self-similarity* of Y in distribution:

$$Y \stackrel{\mathsf{d}}{=} 1 + \xi Y, \tag{7}$$

where ξ is independent of Y. We remark that infinite polynomials similar to Y were studied by [Vervaat (1979)] and many others. There some of the ideas discussed below have already appeared.

A standard application of the strong law of large numbers gives a condition for having an a.s. finite limit Y here, see Theorem 1 in [Székely, G. (1975)]. Namely, when $\mathbf{E}(|\log \xi_j|) < \infty$, one has $Y_n \nearrow Y < \infty$ a.s. if and only if $\mathbf{E}(\log \xi_j) < 0$.

In the special case when Y is defined as in (4), but S_n is the partial sum of an *arbitrary* i.i.d. sequence $(X_j)_{j=1}^{\infty}$ with zero expectation, $Y < \infty$ a.s. iff the drift added is negative: $\nu > 0$. Hence this condition is always assumed in our basic example (simple, symmetric RW).

Next we want to show that self-similarity (7) implies a simple functional equation for the distribution function $F(y) = \mathbf{P}(Y \leq y)$, $y \in \mathbb{R}$. For a modest generalization of our basic case, let us introduce some notations. In (5) let ξ_j take the positive values $\gamma_1 < \cdots < \gamma_N$, and let $p_i = \mathbf{P}(\xi = \gamma_i)$. (In our basic case $N = 2$, $\gamma_1 = e^{-1-\nu}$, $\gamma_2 = e^{1-\nu}$, $p_1 = p_2 = \frac{1}{2}$.) Consider the following similarity transformations: $T_i(x) = \gamma_i x + 1$ $(1 \leq i \leq N)$. When

$$\mathbf{E}(\log \xi) = \sum_{i=1}^{N} p_i \, \log \gamma_i < 0$$

holds, by (7) we have $\mathbf{P}(Y \leq y) = \mathbf{P}(1 + \xi Y \leq y) = \sum_{i=1}^{N} p_i \, \mathbf{P}(1 + \gamma_i Y \leq y | \xi = \gamma_i) = \sum_{i=1}^{N} p_i \, \mathbf{P}(1 + \gamma_i Y \leq y)$. Thus one obtains the following functional equation for the distribution function of Y:

$$F(y) = \sum_{i=1}^{N} p_i \, F(T_i^{-1}(y)). \tag{8}$$

An important special case is when $\gamma_N < 1$ (in the basic case: $\nu > 1$). Then by (5), Y is a bounded random variable. Moreover, each mapping T_i is a contraction, having a unique fixpoint $y_i = (1 - \gamma_i)^{-1}$, $0 < y_1 < \cdots < y_N < \infty$.

Since each T_i is an increasing function, $T_i(y_j) < T_i(y_k)$ if $j < k$. Also, $T_j(y_i) < T_k(y_i)$ if $j < k$. Then it follows that each T_i maps the fundamental interval $I = [y_1, y_N]$ into itself. Clearly, I contains the range of Y too.

In the case $\gamma_N < 1$ it is useful to rephrase the given problem in the language of fractal theory, see e.g. [Falconer (1990)]. Let us introduce the symbolic space $\Sigma = \{\underline{i} = (i_1, i_2, \dots) : i_j = 1, \dots, N\}$, endowed with the countable power of the discrete measure $(p_1, \dots, p_N)$, denoted by $\mathbf{P}$. By (5),

$$Y_n = 1 + \gamma_{i_1}(1 + \gamma_{i_2}(\cdots(1 + \gamma_{i_n}))) = (T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n})(1), \tag{9}$$

with probability $p_{i_1} p_{i_2} \cdots p_{i_n}$. Thus the canonical projection $\Pi : \Sigma \to I$, $\Pi(\underline{i}) = \lim_{k\to\infty}(T_{i_1} \circ \cdots \circ T_{i_k})(1) = \lim_{k\to\infty}(1 + \gamma_{i_1} + \cdots + \gamma_{i_1} \dots \gamma_{i_k})$ maps Σ onto the range of Y. The *attractor* Λ of the iterated function scheme of similarity transformations $(T_1, \dots, T_N)$ is defined as

$$\Lambda = \bigcap_{k\geq 0} \bigcup_{1\leq i_1,\dots,i_k\leq N} (T_{i_1} \circ \cdots \circ T_{i_k})(I), \qquad \Lambda = \bigcup_{i=1}^{N} T_i(\Lambda). \tag{10}$$

Then Λ is a non-empty, compact, self-similar set. In (10) the fundamental interval $I = [y_1, y_N]$ can be replaced by any interval J which is mapped into itself by each T_i, e.g. by $J = [0, y_N] \ni 1$. Thus range$(Y) \subset \Lambda$, cf. (9). The converse is also true, since for any $y \in \Lambda$ and for any $\epsilon > 0$ there is an $\underline{i} \in \Sigma$ and a large enough k such that $y \in (T_{i_1} \circ \cdots \circ T_{i_k})(J)$ and the length $|(T_{i_1} \circ \cdots \circ T_{i_k})(J)| = \gamma_{i_1} \cdots \gamma_{i_k} |J| < \epsilon$. Hence, by (9), $y \in$ range(Y), that is, $\Lambda =$range(Y). Also, the distribution of Y on the real line, which will be denoted by $\mathbf{P}_Y$, is simply $\mathbf{P} \circ \Pi^{-1}$.

We are going to use the notations

$$I_{i_1\dots i_k} = [y_{i_1\dots i_k 1}, y_{i_1\dots i_k N}] = (T_{i_1} \circ \cdots \circ T_{i_k})(I), \tag{11}$$

$y_{i_1\dots i_k l} = (T_{i_1} \circ \cdots \circ T_{i_k})(y_l)$ $(l = 1, \dots, N)$ as well, where $i_j = 1, \dots N$ and $y_l = (1 - \gamma_l)^{-1}$. The length of such an interval is $|I_{i_1\dots i_k}| = \gamma_{i_1} \cdots \gamma_{i_k} |I|$, where $|I| = y_N - y_1$.

Returning to the distribution of Y in the basic case, consider first when the intervals $I_1 = T_1(I) = [y_{11}, y_{12}] = [y_1, y_{12}]$ and $I_2 = T_2(I) = [y_{21}, y_{22}] = [y_{21}, y_2]$ do not overlap, where $y_{12} = 1 + \gamma_1(1 - \gamma_2)^{-1}$, $y_{21} = 1 + \gamma_2(1 - \gamma_1)^{-1}$. Thus there is no overlap iff $y_{12} < y_{21}$, i.e., $\nu > \log(e + e^{-1}) \approx 1.127$. Since $F(y) = 0$ if $y < y_1$ and $F(y) = 1$ if $y \geq y_2$, in this non-overlapping case (8) simplifies as

$$F(y) = \begin{cases} \frac{1}{2} F(T_1^{-1}(y)) & \text{if } y \in [y_1, y_{12}), \\ \frac{1}{2} & \text{if } y \in [y_{12}, y_{21}), \\ \frac{1}{2} + \frac{1}{2} F(T_2^{-1}(y)) & \text{if } y \in [y_{21}, y_2). \end{cases} \tag{12}$$

By the similarities given by T_1 and T_2, applied to (12), one obtains that F has constant value $\frac{1}{4}$ over the interval $[y_{112}, y_{121})$ and constant value $\frac{3}{4}$ on $[y_{212}, y_{221})$. Continuing this way by induction one gets that F has constant dyadic values over such *plateau* intervals:

$$F(y) = 2^{-k-1} + \sum_{j=1}^{k} (i_j - 1) 2^{-j}, \qquad y \in [y_{i_1\dots i_k 12}, y_{i_1\dots i_k 21}), \qquad i_j = 1, 2. \tag{13}$$

The sum of the lengths of these plateaus is $|I|\,(1-(\gamma_1+\gamma_2))\,(1+(\gamma_1+\gamma_2)$ $+(\gamma_1+\gamma_2)^2\;+\cdots)$, so add up to $|I|$. Hence the attractor Λ (the range of Y), i.e., the set of points of increase of F, has zero Lebesgue measure. So it is a Cantor-type set: an uncountable, perfect set of Lebesgue measure zero.

The distribution function F is clearly a continuous singular function. For, if $y_0 \in \Lambda$ and $\epsilon > 0$ is given, take k so that $2^{-k} < \epsilon$. By the construction of Λ, there exists an interval $I_{i_1\ldots i_k} \ni y_0$. Let the left endpoint of the left neighbor plateau of $I_{i_1\ldots i_k}$ be η_1 (or $-\infty$), and the right endpoint of the right neighbor plateau be η_2 (or ∞). If $\delta = \min(y_0-\eta_1, \eta_2-y_0) > 0$, then for any y such that $|y-y_0| < \delta$ one has $|F(y)-F(y_0)| \le 2^{-k} < \epsilon$ by (13).

It is not difficult to see, cf. [Grincevičius (1974)], that in general, any solution of (7) has either absolutely continuous or continuous singular distribution.

We mention that standard results of fractal theory, see Theorem 9.3 in [Falconer (1990)], imply that the Hausdorff dimension s of Λ equals the (unique) solution of the equation $\gamma_1^s + \gamma_2^s = 1$. Solving this equation for ν, we get $\nu = s^{-1}\log(e^s+e^{-s})$. Hence the fractal dimension s is a strictly decreasing function of $\nu > \log(e+e^{-1})$, tending to 1 as $\nu \to \log(e+e^{-1})$ and converging to 0 as $\nu \to \infty$. Also, the Hausdorff measure of Λ is $\mathcal{H}^s(\Lambda) = |I|^s$, where the Hausdorff dimension s is the one defined above. It means that

$$\mathcal{H}^s(\Lambda) = \left(\left(1-e(2\cosh s)^{-1/s}\right)^{-1} - \left(1-e^{-1}(2\cosh s)^{-1/s}\right)^{-1}\right)^s .$$

Thus $\mathcal{H}^s(\Lambda) \to e^2 - e^{-2}$ as $\nu \to \log(e+e^{-1})$ and $\mathcal{H}^s(\Lambda) \to 0$ as $\nu \to \infty$.

Next we are going to show that the distribution of Y is singular w.r.t. Lebesgue measure even in the overlapping case if $\nu > 1$. Again, we consider the slight generalization introduced above. The proof below is based on [Simon, K. et al. (2001)] and on personal communication with K. Simon.

Theorem 1. *Let ξ take the values γ_i $(i = 1,\ldots,N)$, $0 < \gamma_1 < \cdots < \gamma_N < 1$, and let $p_i = \mathbf{P}(\xi = \gamma_i)$. Take an i.i.d. sequence $(\xi_j)_{j=1}^{\infty}$, $\xi_j \overset{\mathsf{d}}{=} \xi$. Then the distribution of $Y = 1 + \xi_1 + \xi_1\xi_2 + \cdots$ is singular w.r.t. Lebesgue measure, if*

$$-\chi_{\mathbf{P}} = \mathbf{E}(\log\xi) = \sum_{i=1}^{N} p_i \log\gamma_i < \sum_{i=1}^{N} p_i \log p_i = -H_{\mathbf{P}}.$$

This will be called the entropy condition. Here $\chi_{\mathbf{P}}$ is the Lyapunov exponent of the iterated function scheme $(T_1,\ldots,T_N)$ corresponding to the Bernoulli measure $\mathbf{P}$.

Proof. We are going to use the fractal theoretical approach and notations introduced above.

We want to show that

$$(\bar{D}\mathbf{P}_Y)(x) = \limsup_{r\searrow 0} \frac{\mathbf{P}_Y(B(x,r))}{\lambda(B(x,r))} = \infty \qquad \mathbf{P}_Y\,\text{a.s.}, \tag{14}$$

where $B(x,r)$ denotes the open ball (in the real line) with center at x and radius r and λ is Lebesgue measure. The statement of the theorem easily follows from this. For, take the set $E = \{x \in I : (\bar{D}\mathbf{P}_Y)(x) = \infty\}$. Then (14) implies that $\mathbf{P}_Y(E) = 1$, while e.g. Theorem 8.6 in [Rudin (1970)] shows that the symmetric derivative $D\mathbf{P}_Y$ exists and is finite λ a.e., so $\lambda(E) = 0$.

Introduce the notation $a_k^{(j)}(\underline{i}) = \#\{l : i_l = j, 1 \le l \le k\}$. Thus

$$\Pi(\underline{i}) = 1 + \sum_{k=1}^{\infty} \prod_{j=1}^{N} \gamma_j^{a_k^{(j)}(\underline{i})}.$$

By the SLLN, the set $A_j = \left\{\underline{i} \in \Sigma : k^{-1} a_k^{(j)}(\underline{i}) \to p_j\right\}$ has probability 1 for every $j = 1, \dots, N$ and so has $A = \bigcap_{j=1}^{N} A_j$. Let $C = \{x \in I : \Pi^{-1}(x) \cap A \ne \emptyset\}$. Then $\mathbf{P}_Y(C) = 1$.

If $x \in C$, there exists $\underline{i} \in A$ such that $\Pi(\underline{i}) = x$ and $k^{-1} a_k^{(j)}(\underline{i}) \to p_j$ as $k \to \infty$ for all $j = 1, \dots, N$. Fix such an $\underline{i}$ and x. Let r_k be the smallest radius such that $B(x, r_k) \supset I_{i_1 \dots i_k}$, where $\underline{i} = (i_1, \dots, i_k, \dots)$ and $I_{i_1 \dots i_k}$ is defined by (11).

The following facts are clear: (a) $x \in \Lambda$, moreover, $x \in I_{i_1 \dots i_k}$, see (10) and (11); (b) $\frac{1}{2}|I_{i_1 \dots i_k}| < r_k \le c|I_{i_1 \dots i_k}|$, where $c > 1$ is arbitrary; (c) $|I_{i_1 \dots i_k}| = |I| \prod_{j=1}^{N} \gamma_j^{a_k^{(j)}(\underline{i})}$; (d) $\mathbf{P}_Y(B(x, r_k)) \ge \mathbf{P}_Y(I_{i_1 \dots i_k}) = \mathbf{P}(i_1, \dots, i_k) = \prod_{j=1}^{N} p_j^{a_k^{(j)}(\underline{i})}$. Using these facts it follows for any $k \ge 1$ that

$$\frac{\mathbf{P}_Y(B(x, r_k))}{\lambda(B(x, r_k))} \ge (2c|I|)^{-1} \left(\frac{\prod_{j=1}^{N} p_j^{k^{-1} a_k^{(j)}(\underline{i})}}{\prod_{j=1}^{N} \gamma_j^{k^{-1} a_k^{(j)}(\underline{i})}} \right)^k .$$

By our assumptions concerning x and $\underline{i}$, the ratio on the right hand side converges to $(p_1^{p_1} \cdots p_N^{p_N}) / (\gamma_1^{p_1} \cdots \gamma_N^{p_N})$ as $k \to \infty$. The entropy condition of the theorem implies that this latter ratio is larger than 1. Hence (14) holds, and this completes the proof. □

Returning to our basic case, consider the entropy condition when $\gamma_1 = e^{-1-\nu}$, $\gamma_2 = e^{1-\nu}$ and $p_1 = p_2 = \frac{1}{2}$. The condition holds iff $\nu > \log 2 \approx 0.693$, since this is equivalent to $\frac{1}{2}(-1-\nu) + \frac{1}{2}(1-\nu) < \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}$. Combining this with the condition $\gamma_2 < 1$, this means that the distribution of Y is singular w.r.t. Lebesgue measure for any $\nu > 1$.

Characterization of the distribution of Y when $0 < \nu \le 1$ remains open. In that case one of the two similarity mappings, T_2, is not a contraction anymore, and that situation requires more sophisticated tools than the ones above.

3 The moments of the discrete exponential functional

Let us consider first the general case: $(\xi_j)_{j=1}^{\infty}$ i.i.d., $\xi_j > 0$, as at the beginning of the previous section. Now we turn our attention to the moments of $Y = 1 + \xi_1 + \xi_1\xi_2 + \cdots$. If Y_n is defined by (5), the equality in law (6) implies

$$\mathbf{E}(Y_n^p) = \mathbf{E}\left((1 + \xi Y_{n-1})^p\right) = \sum_{k=0}^{p} \binom{p}{k} \mu_k \mathbf{E}(Y_{n-1}^k), \tag{15}$$

where $p \ge 0$ integer and $\mu_k = \mathbf{E}(\xi^k)$. As $n \to \infty$, by monotone convergence we obtain

$$\mathbf{E}(Y^p) = \sum_{k=0}^{p} \binom{p}{k} \mu_k \mathbf{E}(Y^k). \tag{16}$$

In (15) and (16) both sides are either finite positive, or $+\infty$.

Theorem 2. *Let $(\xi_j)_{j=1}^{\infty}$ be an i.i.d. sequence, $\xi_j > 0$, and $Y = 1 + \xi_1 + \xi_1\xi_2 + \cdots$. For $p \geq 1$ real, $\mathbf{E}(Y^p) < \infty$ if and only if $\mu_p = \mathbf{E}(\xi^p) < 1$. Then $\mu_q < 1$ for any $1 \leq q \leq p$ and $\mathbf{E}(Y^p) \leq (1 - \mu_p^{1/p})^{-p}$ as well. In this case if $p \geq 1$ is an integer, we also have the recursion formula*

$$\mathbf{E}(Y^p) = \frac{1}{1-\mu_p} \sum_{k=0}^{p-1} \binom{p}{k} \mu_k \mathbf{E}(Y^k). \tag{17}$$

Proof. First, (6) and the simple inequality $(1+x)^p \geq 1 + x^p$ ($x \geq 0$, $p \geq 1$, real) imply that $\mathbf{E}(Y_n^p) \geq 1 + \mu_p \mathbf{E}(Y_{n-1}^p)$. Suppose that $\mu_p \geq 1$. Since $\mathbf{E}(Y_0^p) = 1$, taking limit as $n \to \infty$, one gets that $\mathbf{E}(Y^p) = \infty$.

Conversely, suppose that $\mu_p < 1$. Then by Hölder's (or by Jensen's) inequality, $\mu_q \leq \mu_p^{q/p} < 1$ for any $1 \leq q \leq p$ as well. We want to show that $\mathbf{E}(Y^p)$ is finite. Let us begin by observing that $\mathbf{E}\left((Y_j - Y_{j-1})^p\right) = \mathbf{E}\left((\xi_1 \cdots \xi_j)^p\right) = \mu_p^j$ ($j \geq 1$). Hence by the triangle inequality and $Y_0 = 1$ we get that

$$\mathbf{E}(Y_n^p)^{1/p} \leq 1 + \sum_{j=1}^{n} \mathbf{E}\left((Y_j - Y_{j-1})^p\right)^{1/p} = \sum_{j=0}^{n} \mu_p^{j/p} < \frac{1}{1-\mu_p^{1/p}},$$

for any $n \geq 1$ when $\mu_p < 1$. Taking limit as $n \to \infty$, it follows that $\mathbf{E}(Y^p) \leq (1 - \mu_p^{1/p})^{-p} < \infty$.

Finally, the recursion formula (17) directly follows from (16) when $\mu_p < 1$, $p \geq 1$ integer. $\square$

For integer $p \geq 1$ it follows from (17) by induction that $\mathbf{E}(Y^p)$ is a rational function of the moments $\mu_1, \ldots, \mu_p$:

$$\mathbf{E}(Y^p) = \frac{1}{(1-\mu_1)\cdots(1-\mu_p)} \sum_{(j_1,\ldots,j_{p-1}) \in \{0,1\}^{p-1}} a^{(p)}_{j_1,\ldots,j_{p-1}} \mu_1^{j_1} \cdots \mu_{p-1}^{j_{p-1}}, \tag{18}$$

where the coefficients of the numerator are *universal* constants, independent of the distribution of ξ_j.

These universal coefficients $a^{(p)}_{j_1,\ldots,j_{p-1}}$ in the numerator of (18) make a symmetrical, Pascal's triangle-like table if they are ordered according to multiindices $(j_{p-1}, \ldots, j_1)$ as binary numbers, see the rows $p = 1, \ldots, 5$:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 2 \quad 1 \\
1 \quad 3 \quad 5 \quad 3 \quad 3 \quad 5 \quad 3 \quad 1 \\
1 \quad 4 \quad 9 \quad 6 \quad 9 \quad 16 \quad 11 \quad 4 \quad 4 \quad 11 \quad 16 \quad 9 \quad 6 \quad 9 \quad 4 \quad 1
\end{array}$$

Rather interestingly, based on the above table, one may conjecture that each coefficient $a^{(p)}_{j_1,\ldots,j_{p-1}}$ is equal to the number of permutations π of the set $\{1, \ldots, p\}$ which have descent $\pi(i) > \pi(i+1)$ exactly where $j_i = 1$, $1 \leq i \leq p-1$. Another interesting conjecture is that the following recursion holds:

$$a^{(p+1)}_{i_1,\ldots,i_p} = \sum_{(j_1,\ldots,j_{p-1}) \in S(i_1,\ldots,i_p)} a^{(p)}_{j_1,\ldots,j_{p-1}}, \qquad (i_1, \ldots, i_p) \in \{0,1\}^p, \qquad p \geq 1,$$

where $S(i_1, \dots, i_p)$ is the set of all distinct binary sequences obtained from $(i_1, \dots, i_p)$ by deleting exactly one digit; $a^{(1)} = 1$. For example, $a^{(5)}_{0110} = 11 = a^{(4)}_{110} + a^{(4)}_{010} + a^{(4)}_{011}$. This recursion would imply that the above table contains only positive integers and has the symmetries $a^{(p+1)}_{i_1,\dots,i_p} = a^{(p+1)}_{i_p,\dots,i_1} = a^{(p+1)}_{1-i_1,\dots,1-i_p} = a^{(p+1)}_{1-i_p,\dots,1-i_1}$.

There is a nice analogy between the moments of the exponential functional of a subordinator and the moments of Y, compare (3) and (18). First, the sum of the coefficients in the numerator of (18) is $p!$, as can be seen by induction. For, if one explicitly writes down $\mathbf{E}(Y^p)$, based on the recursion (17), taking a common denominator, the numerator of each earlier term except the last one is multiplied by factors $1 - \mu_k$. In the sum of the coefficients of the numerator it means a multiplication by zero. On the other hand, in the last term one multiplies the numerator of $\mathbf{E}(Y^{p-1})$ by $p\mu_{p-1}$, which results the sum $p!$ of the coefficients by the induction.

Second, there is a relationship between the denominators of (3) and (18) as well. In the special case when Y is defined as in (4), but S_n is the partial sum of an arbitrary i.i.d. sequence $(X_j)_{j=1}^{\infty}$ with zero expectation, $\Phi(\lambda) = -n^{-1} \log \mathbf{E}\left(\exp(\lambda(S_n - \nu n))\right) = -\log \mathbf{E}(\xi^\lambda)$, so $\Phi(k) = -\log \mu_k$, corresponding to the factors in the denominator of (3). The factors $1 - \mu_k$ in the denominator of (18) are tangents to these.

Finally, let us consider the moments of Y in our *basic case*. Then $\mu_k = \mathbf{E}(\xi^k) = \exp(-k\nu)\cosh(k)$. Since $\cosh(k) < e^k$ for any $k > 0$, it follows that $\mu_k < 1$ for any $k \geq 1$ when $\nu \geq 1$, therefore all positive integer moments of Y are finite in this case by Theorem 2. In particular, in Section 2 we saw that Y is a bounded random variable if $\nu > 1$, hence the positive integer moments characterize its distribution. On the other hand, when $0 < \nu < 1$, only finitely many moments of Y are finite. For, $\mu_k \geq 1$ if $0 < \nu \leq k^{-1} \log\cosh(k) \nearrow 1$ as $k \to \infty$. For example, even $\mu_1 \geq 1$ (and consequently all $\mathbf{E}(Y^p) = \infty$) if $0 < \nu < \log\cosh(1) \approx 0.43378$.

4 Approximation of the exponential functional of BM

In this final section we are going to show that taking a suitable nested sequence of RWs the resulting sequence of discrete exponential functionals (4) converges almost surely to the corresponding exponential functional $\mathcal{I}$ of BM. Based on this, using convergence of moments, we will give an elementary proof of theorem (1) of Dufresne and Yor.

The underlying RW construction of BM was first introduced by [Knight (1962)], and simplified and somewhat improved by [Révész (1990)] and [Szabados (1996)]. This construction starting from an independent sequence of RWs $(S_m(k), k \geq 0)_{m=0}^{\infty}$, constructs a dependent sequence of RWs $(\widetilde{S}_m(k), k \geq 0)_{m=0}^{\infty}$ by "twisting" so that the shrunken and linearly interpolated sequence $(B_m(t) = 2^{-m} \widetilde{S}_m(t2^{2m}), t \geq 0)_{m=0}^{\infty}$ a.s. converges to BM $(B(t), t \geq 0)$, uniformly on bounded intervals, see Theorem 3 in [Szabados (1996)].

We need one more result about this approximation here. This is stated in a somewhat sharper form than in the cited reference, but can be read easily from

the proof there. Namely, see Lemma 4 in [Szabados (1996)], for almost every ω there exists an $m_0(\omega)$ such that for any $m \geq m_0(\omega)$ and for any $K \geq e$, one has

$$\sup_{j\geq 1} \sup_{0\leq t\leq K} |B_{m+j}(t) - B_m(t)| \leq K^{\frac{1}{4}} (\log K)^{\frac{3}{4}} m 2^{-\frac{m}{2}}. \tag{19}$$

Lemma 1. *Let $B_m(t) = 2^{-m}\widetilde{S}_m(t2^{2m})$, $t \geq 0$, $m \geq 0$, be a sequence of shrunken simple symmetric RWs that a.s. converges to BM $(B(t), t \geq 0)$, uniformly on bounded intervals. Then for any $\nu > 0$, as $m \to \infty$,*

$$\begin{aligned} Y_m &= 2^{-2m} \sum_{k=0}^{\infty} \exp\left(2^{-m}\widetilde{S}_m(k) - \nu k 2^{-2m}\right) \\ \to \quad \mathcal{I} &= \int_0^\infty \exp\left(B(t) - \nu t\right)\, dt < \infty \qquad \textit{a.s.} \end{aligned}$$

Proof. The basic idea of the proof is that the sequence of functions $f_m(t, \omega) = \exp\left(B_m(t) - \nu t\right)$, converges to $f(t, \omega) = \exp\left(B(t) - \nu t\right)$ for $t \in [0, \infty)$ as $m \to \infty$, for almost every ω. If one can find a function $g(t, \omega) \in L^1[0, \infty)$, that dominates each f_m for $m \geq m_0(\omega)$, then their integrals on $[0, \infty)$ also converge to the integral of f, and then we are practically done.

First, by (19), for a.e. ω there exists an $m_0 = m_0(\omega)$ so that for any $K \geq e$,

$$\sup_{m\geq m_0} \sup_{0\leq t\leq K} |B_m(t) - B_{m_0}(t)| \leq K^{\frac{1}{4}} (\log K)^{\frac{3}{4}} \leq K^{\frac{1}{2}} \log K, \tag{20}$$

where we supposed that m_0 was chosen large enough so that $m_0 2^{-m_0/2} \leq 1$.

Second, by the law of iterated logarithms,

$$\limsup_{t\to\infty} \frac{B_{m_0}(t)}{(2t \log\log t)^{\frac{1}{2}}} = \limsup_{u\to\infty} \frac{\widetilde{S}_{m_0}(u)}{(2u \log\log u)^{\frac{1}{2}}} = 1 \qquad \text{a.s.},$$

where $u = t2^{2m_0}$. Hence for a.e. ω, there is a $K_0 = K_0(\nu, \omega)$, such that for any $t \geq K_0$,

$$B_{m_0}(t) \leq 2(t \log\log t)^{\frac{1}{2}} \leq 2t^{\frac{1}{2}} \log t, \tag{21}$$

where K_0 is chosen so large that $3t^{\frac{1}{2}} \log t \leq \nu t/2$ for any $t \geq K_0$.

Since a.s. any path of B_{m_0} is continuous, it is bounded on the interval $[0, K_0]$. Then by (20), we have an upper bound uniform in m: for any $m \geq m_0$ and $t \in [0, K_0]$, $B_m(t) \leq M(\omega)$. On the other hand, when $t > K_0$, by (21), $B_{m_0}(t) \leq 2t^{\frac{1}{2}} \log t$ and so by (20), $B_m(t) \leq 3t^{\frac{1}{2}} \log t$, for any $m \geq m_0$.

Summarizing, the function

$$g(t, \omega) = \begin{cases} e^{M(\omega)} & \text{if } 0 \leq t \leq K_0(\nu, \omega), \\ e^{-\nu t/2} & \text{if } t > K_0(\nu, \omega), \end{cases}$$

is an integrable function on $[0, \infty)$, dominating $\exp(B_m(t) - \nu t)$ for each $m \geq m_0(\omega)$. This implies that

$$\lim_{m\to\infty} \int_0^\infty \exp\left(B_m(t) - \nu t\right)\, dt = \int_0^\infty \exp\left(B(t) - \nu t\right)\, dt < \infty \qquad \text{a.s.}$$

Finally, compare $\int_0^\infty \exp\left(B_m(t) - \nu t\right)\, dt$ to $Y_m = 2^{-2m} \sum_{k=0}^\infty \exp\big(B_m(k2^{-2m})$ $-\nu k 2^{-2m}\big)$ that appears in the statement of the lemma. Applying the uniform

domination of $\exp(B_m(t) - \nu t)$ by the function g shown above, both the tail of the integral on the interval $[K_0, \infty)$ and the tail of the sum for $k \geq \lceil K_0 2^{2m} \rceil$ is smaller than $\int_{K_0}^{\infty} \exp(-\nu t/2)\, dt$, thus their difference is uniformly arbitrarily small for any $m \geq m_0$ if K_0 is large enough. On the interval $[0, K_0]$ the difference of the integral and the sum (which is a Riemann sum of a continuous function) tends to zero uniformly as $m \to \infty$, since on each subinterval of length 2^{-2m}, the difference of $B_m(t)$ and $B_m(k2^{-2m})$ is at most 2^{-m}. This completes the proof of the lemma. $\square$

Next we want to apply the results of the previous sections to Y_m. To do this we introduce the following notations. For $m \geq 0$ and $n \geq 1$ let

$$\begin{aligned} Y_{m,n} &= 2^{-2m} \sum_{k=0}^{n} \exp\left(2^{-m} \widetilde{S}_m(k) - \nu k 2^{-2m}\right) \\ &= 2^{-2m} \left(1 + \xi_{m1} + \xi_{m1}\xi_{m2} + \xi_{m1} \cdots \xi_{mn}\right) \end{aligned} \tag{22}$$

and $Y_{m,0} = 2^{-2m}$, where $\xi_{mj} = \exp(2^{-m}\widetilde{X}_m(j) - \nu 2^{-2m})$. Here $\widetilde{X}_m(j) = \widetilde{S}_m(j) - \widetilde{S}_m(j-1)$ $(j = 1, 2, \dots)$ is an i.i.d. sequence, $\mathbf{P}(\widetilde{X}_m(j) = \pm 1) = \frac{1}{2}$.

Then $Y_{m,n} \nearrow Y_m$ as $n \to \infty$, $Y_m < \infty$ a.s. iff $\nu > 0$, and Y_m satisfies the following self-similarity in distribution:

$$Y_m \stackrel{\mathsf{d}}{=} 2^{-2m} + \xi_m Y_m \qquad \text{or} \qquad 2^{2m} Y_m \stackrel{\mathsf{d}}{=} 1 + \xi_m 2^{2m} Y_m, \tag{23}$$

where ξ_m and Y_m are independent, $\xi_m \stackrel{\mathsf{d}}{=} \xi_{mj}$. Using the notations of Section 2, now $\gamma_1 = \exp(-2^{-m} - \nu 2^{-2m})$, $\gamma_2 = \exp(2^{-m} - \nu 2^{-2m})$, $p_1 = p_2 = \frac{1}{2}$. If $\nu > 2^m$, $\gamma_2 < 1$ holds, so the similarity transformations T_1 and T_2 are contractions, mapping the interval $I = [(1-\gamma_1)^{-1}, (1-\gamma_2)^{-1}]$ into itself. By Theorem 1, the distribution of Y_m is singular w.r.t. Lebesgue measure if $\nu > 2^{2m} \log 2$ $(m \geq 1)$. Moreover, there is no overlap in the ranges of T_1 and T_2 iff $\nu > 2^{2m} \log(2\cosh(2^{-m}))$. As $m \to \infty$ this means asymptotically that $\nu > \frac{1}{2} + 2^{2m} \log 2 + o(1)$.

For $m \geq 0$ and k integer let

$$\mu_{mk} = \mathbf{E}(\xi_m^k) = \exp(-\nu k 2^{-2m}) \cosh(k 2^{-m}).$$

Since Theorem 2 is applicable to $2^{2m} Y_m$, one obtains that $\mathbf{E}(Y_m^p) < \infty$ if and only if $\mu_{mp} < 1$ and then the following recursion is valid for $p \geq 1$ integer:

$$\mathbf{E}(Y_m^p) = \frac{1}{1 - \mu_{mp}} \sum_{k=0}^{p-1} \binom{p}{k} 2^{-2m(p-k)} \mu_{mk} \mathbf{E}(Y_m^k). \tag{24}$$

Now using $\cosh(x) < e^x$ $(x > 0)$, it follows that $\mu_{mk} < \exp(k 2^{-m}(1 - \nu 2^{-m}))$. So $\mu_{mk} < 1$ for any $k \geq 1$ if $\nu \geq 2^m$. If $0 < \nu < 2^m$, only finitely many positive moments are finite, since $\mu_{mk} \geq 1$ when $0 < \nu < 2^{2m} k^{-1} \log\cosh(k 2^{-m}) \to 2^m$ as $k \to \infty$.

More importantly,

$$\mu_{mk} < \exp\left(k 2^{-2m} \left(\frac{k}{2} - \nu\right)\right) \qquad (k \geq 1), \tag{25}$$

since $\cosh(x) < \exp(x^2/2)$ when $x > 0$ (compare the Taylor series). Thus $\mu_{mk} < 1$ for any $m \geq 0$ if $\nu \geq \frac{k}{2}$. This condition is sharp as $m \to \infty$. For, apply $e^x = 1 + x + o(x)$ and $\cosh(x) = 1 + x^2/2 + o(x^2)$ (as $x \to 0$) to the definition of μ_{mk}. Then

$$\mu_{mk} = 1 + k2^{-2m}\left(\frac{k}{2} - \nu\right) + o(2^{-2m}), \tag{26}$$

for any fixed k as $m \to \infty$.

Lemma 2. *If p is a positive integer such that $\frac{p}{2} < \nu$, then*

$$\lim_{m\to\infty} \mathbf{E}(Y_m^p) = \frac{1}{\prod_{k=1}^{p}\left(\nu - \frac{k}{2}\right)} < \infty. \tag{27}$$

Proof. By (25), for any positive integer p such that $\frac{p}{2} < \nu$ we have $\mu_{mp} < 1$. Since Theorem 2 is valid for $2^{2m}Y_m$, the recursion formula (24) holds, and by induction one gets $\mathbf{E}(Y_m^p)$ as a rational function of the moments $\mu_{m1}, \ldots, \mu_{mp}$, similarly to formula (18). The argument below formula (18) also applies here too, showing that the sum of the coefficients in the numerator of this rational function is $p!2^{-2mp}$. The extra factor comes from the difference that Y_m is multiplied by 2^{2m} here, compare equations (17) and (24). Since each $\mu_{mk} \to 1$ as $m \to \infty$, it follows that 2^{2mp} times the numerator tends to $p!$.

By (26), we get that $1 - \mu_{mk} = k2^{-2m}(\nu - \frac{k}{2}) + o(2^{-2m})$ if k is fixed and $m \to \infty$. So 2^{2mp} times the denominator of the rational function tends to $p!\prod_{k=1}^{p}(\nu - \frac{k}{2})$ as $m \to \infty$. This and the limit of the numerator together imply the statement of the lemma. $\square$

Our next objective is to give an asymptotic formula, similar to (27), for the negative moments of Y_m as $m \to \infty$.

Lemma 3. *For all integer $p \geq 1$, we have*

$$\lim_{m\to\infty} \mathbf{E}(Y_m^{-p}) = \lim_{m\to\infty} \mathbf{E}(Y_m^{-1}) \prod_{k=1}^{p-1}\left(\nu + \frac{k}{2}\right), \tag{28}$$

where $\lim_{m\to\infty} \mathbf{E}(Y_m^{-1}) < \infty$.

Proof. We want to show (28) by establishing a recursion. Introduce the notations $z_{m,k} = \mathbf{E}(Y_m^{-k})$ and $\mu_{m,-k} = \mathbf{E}(\xi_m^{-k})$ for $k \geq 1$ integer. By (22), $0 < Y_m^{-1} < 2^{2m}$, hence all negative moments $z_{m,k}$ of Y_m are finite.

The self-similarity equation (23) implies that $\xi_m Y_m \stackrel{\mathsf{d}}{=} Y_m - 2^{-2m}$ and so

$$\xi_m^{-1} Y_m^{-1} \stackrel{\mathsf{d}}{=} \frac{Y_m^{-1}}{1 - 2^{-2m}Y_m^{-1}},$$

where ξ_m and Y_m are independent. Taking kth moment ($k \geq 1$ integer) on both sides and applying the Taylor series

$$\frac{x^k}{(1-x)^k} = \sum_{n=k}^{\infty}\binom{n-1}{k-1}x^n,$$

valid for any $|x| < 1$, one obtains

$$\mu_{m,-k}\, z_{m,k} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} 2^{-2m(n-k)} z_{m,n},$$

with the notations introduced above. This implies that

$$(\mu_{m,-k} - 1) z_{m,k} - k2^{-2m} z_{m,k+1} = a(m,k), \tag{29}$$

where

$$a(m,k) = \sum_{n=k+2}^{\infty} \binom{n-1}{k-1} 2^{-2m(n-k)} z_{m,n} \geq 0.$$

Next we want to give an upper bound for $a(m,k)$, which goes to zero fast enough as $m \to \infty$. Since $\xi_m \geq \gamma_1 = \exp(-2^{-m} - \nu 2^{-2m})$, by (22) it follows that

$$Y_m^{-1} \leq 2^{2m} \left(\sum_{j=0}^{\infty} \gamma_1^j \right)^{-1} = 2^{2m}(1-\gamma_1) \leq 2^{2m}(2^{-m} + \nu 2^{-2m}) \leq 2^{m+1},$$

if $m \geq \log(\nu)/\log(2)$, where we used that $1 - e^{-x} \leq x$, for any real x. This implies that $z_{m,r+j} = \mathbf{E}(Y_m^{-r-j}) \leq \mathbf{E}(Y_m^{-r}) \left(\sup(Y_m^{-1})\right)^j \leq z_{m,r} 2^{(m+1)j}$ for $r, j \geq 0$. Substituting this into the definition of $a(m,k)$, one gets that

$$\begin{aligned} a(m,k) &\leq z_{m,k+1} \sum_{n=k+2}^{\infty} \binom{n-1}{k-1} 2^{-2m(n-k)} 2^{(m+1)(n-k-1)} \\ &= z_{m,k+1} 2^{k(m-1)-m-1} \sum_{n=k+2}^{\infty} \binom{n-1}{k-1} 2^{(1-m)n} \\ &= z_{m,k+1} 2^{-m-1} \left((1-2^{1-m})^{-k} - 1 - k2^{1-m} \right) \\ &\leq z_{m,k+1} 2^{-m-1} 4k(k+1) 2^{-2m} = z_{m,k+1} 2k(k+1) 2^{-3m}, \end{aligned}$$

if m is large enough, depending on k.

Let us substitute this estimate of $a(m,k)$ into (29) and express the following ratio:

$$\frac{z_{m,k+1}}{z_{m,k}} = \frac{\mu_{m,-k} - 1}{k2^{-2m}(1 + O(2^{-m}))}.$$

Apply the asymptotics (26) here with $-k$:

$$\frac{z_{m,k+1}}{z_{m,k}} = \frac{\nu + \frac{k}{2} + o(1)}{1 + O(2^{-m})},$$

as $m \to \infty$. This implies the equality $\lim_{m\to\infty} z_{m,k+1}/z_{m,k} = \nu + \frac{k}{2}$, and thus for any positive integer p,

$$\lim_{m\to\infty} \frac{\mathbf{E}(Y_m^{-p})}{\mathbf{E}(Y_m^{-1})} = \prod_{k=1}^{p-1} \left(\nu + \frac{k}{2} \right). \tag{30}$$

It remains to show that $\mathbf{E}(Y_m^{-1})$ has a finite limit as $m \to \infty$. Writing $Y_m^{-2} = Y_m Y_m^{-3}$ and applying the Cauchy-Schwarz inequality, one obtains $\left(\mathbf{E}(Y_m^{-2})\right)^2 \leq$

$\mathbf{E}(Y_m^2)\mathbf{E}(Y_m^{-6})$, or $\mathbf{E}(Y_m^{-2}) \le \mathbf{E}(Y_m^2)\mathbf{E}(Y_m^{-6})/\mathbf{E}(Y_m^{-2})$. Suppose first that $\nu > 1$ and take limit here on the right hand side as $m \to \infty$, applying (27) and (30). It follows that $\sup_{m \ge 1} \mathbf{E}(Y_m^{-2}) \le \infty$. As $\mathbf{E}(Y_m^{-2})$ is an increasing function of ν by its definition, hence the same is true for any $\nu \in (0, 1]$ as well. Since by Lemma 1, $Y_m \to \mathcal{I}$ a.s., where each Y_m and also $\mathcal{I}$ take values in $(0, \infty)$ a.s., it follows that $Y_m^{-1} \to I^{-1}$ a.s. Then by the L^2 uniform boundedness of Y_m^{-1} $(m \ge 0)$ shown above, $\mathbf{E}(Y_m^{-1}) \to \mathbf{E}(I^{-1}) < \infty$ follows as well. This ends the proof of the lemma. □

Finally, it turns out that Y_m^{-1} converges to $\mathcal{I}^{-1}$ in any L^p. This makes it possible to recover the result (1) of [Dufresne (1990)] and [Yor (1992)].

Theorem 3. *Let $B_m(t) = 2^{-m}\widetilde{S}_m(t2^{2m})$, $t \ge 0$, $m \ge 0$, be a sequence of shrunken simple symmetric RWs that a.s. converges to BM $(B(t), t \ge 0)$, uniformly on bounded intervals. Take*

$$Y_m = 2^{-2m} \sum_{k=0}^{\infty} \exp\left(B_m(k2^{-2m}) - \nu k 2^{-2m}\right) = 2^{-2m}\left(1 + \xi_{m1} + \xi_{m1}\xi_{m2} + \cdots\right)$$

and

$$\mathcal{I} = \int_0^\infty \exp\left(B(t) - \nu t\right) \, dt$$

when $\nu > 0$. Then the following statements hold true:

(a) Y_m^{-1} converges to $\mathcal{I}^{-1}$ in L^p for any $p \ge 1$ real and $\lim_{m\to\infty} \mathbf{E}(Y_m^{-p}) = \mathbf{E}(\mathcal{I}^{-p}) < \infty$;

(b) $\mathcal{I} \stackrel{\mathrm{d}}{=} 2/Z_{2\nu}$, where $Z_{2\nu}$ is a gamma distributed random variable with index 2ν and parameter 1;

(c) Y_m converges to $\mathcal{I}$ in L^p for any integer p such that $1 \le p < 2\nu$ (supposing $\nu > \frac{1}{2}$) and then $\lim_{m\to\infty} \mathbf{E}(Y_m^p) = \mathbf{E}(\mathcal{I}^p) < \infty$. The same is true for any real q, $1 \le q < p$.

Proof. By Lemma 1, $Y_m \to \mathcal{I}$ a.s., where each Y_m and also $\mathcal{I}$ take values in $(0, \infty)$ a.s. Hence $Y_m^{-1} \to \mathcal{I}^{-1}$ a.s. By Lemma 3, $\lim_{m\to\infty} \mathbf{E}(Y_m^{-k}) < \infty$ for any $k \ge 1$ integer, so (a) follows.

Thus by (a) and Lemma 3, for any integer $p \ge 1$,

$$a_p = \mathbf{E}(\mathcal{I}^{-p}) = c \prod_{k=1}^{p-1} \left(\nu + \frac{k}{2}\right) = c2^{1-p}(2\nu + 1) \cdots (2\nu + p - 1) < \infty,$$

where $c = \mathbf{E}(\mathcal{I}^{-1})$. By a classical result, see [Simon, B. (1998)], a Stieltjes moment problem is determinate, that is the moments uniquely determine a probability distribution on $[0, \infty)$, if there exist constants $C > 0$ and $R > 0$ such that $a_p \le CR^p(2p)!$ for any $p \ge 1$ integer. In the present case $a_p \le c2^{-p}(p+1)!$ when $\nu \le 1$ and $a_p \le c(\nu/2)^{p-1}(p+1)!$ when $\nu > 1$, so the moment problem for $\mathcal{I}^{-1}$ is determinate and it also follows that $\mathcal{I}^{-1}$ has a finite moment generating function in a neighborhood of the origin.

Also, using the moments of the gamma distribution we get

$$b_p = \mathbf{E}(2^{-p} Z_{2\nu}^p) = 2^{-p} \frac{\Gamma(2\nu + p)}{\Gamma(2\nu)} = 2^{-p}(2\nu)(2\nu + 1) \cdots (2\nu + p - 1),$$

for any $p \geq 1$, and $Z_{2\nu}/2$ has a finite moment generating function in a neighborhood of the origin as well. Writing down the two moment generating functions by the help of the moments a_p and b_p, respectively, it follows that

$$\mathbf{E}(\exp(u\mathcal{I}^{-1})) = \frac{c}{\nu}\mathbf{E}(\exp(uZ_{2\nu}/2))$$

in a neighborhood of the origin. Substituting $u = 0$, one obtains $c = \mathbf{E}(\mathcal{I}^{-1}) = \nu$ and this proves (b).

Finally, again, $Y_m \to \mathcal{I}$ a.s. by Lemma 1. If p is an integer such that $1 \leq p < 2\nu$, by Lemma 2, using the moments of the gamma distribution, and by (b), we have $\lim_{m\to\infty} \mathbf{E}(Y_m^p) = \mathbf{E}(2^p Z_{2\nu}^{-p}) = \mathbf{E}(\mathcal{I}^p)$. This proves (c). $\square$

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